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Polarization of the D0 ground state in quantum mechanics and supergravity

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Abstract: The presence of a distant D4-brane is used to further investigate the duality between M-theory and D0-brane quantum mechanics. Although the D4-brane background fields are not strong enough to induce a classical dielectric effect in the D0 system, a polarization of the quantum mechanical ground state does result. A similar deformation arises for the bubble of normal space found near D0-branes in classical supergravity solutions. These deformations are compared and are shown to have the same structure in each case. Brief comments are included on the relation of D0-branes in this background to D0-branes as instantons in the D4-brane field theory and an appendix addresses certain infrared issues associated with 't Hooft scaling in 0+1 dimensions.

Keywords: M-Theory, D-branes, M(atrix) Theories.

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Contents

1. Introduction

In recent years the outlook within string theory has changed immensely. While perturbative string calculations are still of interest, the new cornerstones of the theory are non-perturbative duality conjectures. Some of the most impressive such conjectures are those of matrix theory [1] and Maldacena's AdS/CFT conjecture [2] and its generalizations [3]. Predictions of these conjectures are verified in ever increasing detail, including impressive recent results [4] which receive no prediction from supersymmetry.

These conjectures relate the physics of certain gravitating systems to that of specific non-gravitating gauge theories. The dynamics of the dual field theories are deduced from the low energy effective actions of the various non-abelian D-brane systems (see e.g., [5]). The correspondences appear to rely on the particular form of the non-abelian interactions. Indeed, this form can be traced to important properties such as the large number of light degrees of freedom that account for black hole entropy [6].

Recently, several investigations [7, 8] have uncovered the form of certain non-abelian couplings of D-branes to supergravity background fields. It is natural to assume that the gauge theory/gravity dualities continue to apply when couplings to such backgrounds are included. Our goal here is to investigate this idea in a particular context by studying the 'polarization' of the Dp-brane bound state in the background of a D$(p+4)$-brane. For definiteness, we shall concentrate on the D0/D4 context.

In certain cases, the application of a Ramond-Ramond background field to a D0-brane system induces a classical dielectric effect and causes the D0-branes to deform into a non-commutative D2-brane [8]. While the Ramond-Ramond fields of our D4-brane background will not be strong enough to induce such a classical effect, they do modify the potential that shapes the non-abelian character of the quantum D0 bound state. As a result, this bound state is deformed, or polarized.

Two aspects of this deformation will be studied and compared with the corresponding supergravity system. Fundamental to this comparison will be the connection described by Polchinski [9] relating the size of the matrix theory bound state to the size of the bubble of space that is well-described by classical supergravity in the near D0-brane spacetime. The near D0-brane spacetime is obtained by taking a limit in which open strings decouple from closed strings and the result is a ten-dimensional spacetime which has small curvature and small string coupling when one is reasonably close (though not too close) to the D0-branes. However, if one moves beyond some critical r_c, the curvature reaches the string scale. As a result, the system beyond r_c is not adequately described by the massless fields of classical supergravity. Our goal is therefore to compare the deformations of the non-abelian D0-brane bound state with the deformations of this bubble of 'normal' space around a large stack of D0-branes.

The deformations we study can be thought of as induced by the presence of a background D4-brane. Strictly speaking, what we mean is that we consider an appropriate limit of the D0/D4 system in which the open strings between D0's decouple from closed strings and in which the strength of the D4-brane background is held constant. While the details are difficult to compute, certain scaling behaviors will be deduced below. As has become common in string theory, we find that the quantum mechanical effects of the non-abelian D0-brane couplings correctly reproduce the effects of classical supergravity in the large N limit.

The organization of the paper is straightforward. We address the polarization of the non-abelian ground state in section 2 and of the supergravity bubble in section 3. We then close with some discussion in section 4, including some comments about other Dp/D$(p+4)$ systems. An appendix A includes a more detailed treatment of infrared effects and 't Hooft scaling in 0+1 dimensional hamiltonian perturbation theory. A consequence of this analysis is that it strengthens the argument that Polchinski's upper bound [9] on the size of the D0 bound state in fact gives the complete scaling with N.

2. D0-brane quantum mechanics

We begin with the world-volume effective field theory describing N D0-Branes in the standard D4-brane background. This action is a suitable generalization of the action for a single D0-brane, consisting of the Born-Infeld term together with appropriate Chern-Simon terms. However, the full action encodes the Chan-Paton factors or non-abelian degrees of freedom that arise from strings stretching between the D0-branes. After presenting this effective theory, we specialize to the case of the D4-brane background and study the resulting deformations of the bound state.

2.1 Preliminaries

We will be using the couplings first derived in [12]. In order to see the relation to the abelian case, it is convenient to display the bosonic parts of these couplings using the action proposed by Myers [8]. Fermions can then be added through an appropriate supersymmetrization. While this action contains the relevant terms from [7] and coincides with Tseytlin's proposal [10] in flat space-time, it is known to require corrections at sixth order in the non-abelian field strength of the world-volume gauge field [11]. However, such high orders of accuracy will not be required for our discussion.

The first part of the non-abelian D0 effective action is the Born-Infeld term

$$S_{BI} = -T_0 \int dt \; \mathrm{STr} \left(e^{-\phi} \sqrt{-\left(P\left[E_{ab} + E_{ai}(Q^{-1}-\delta)^{ij} E_{jb}\right]\right) \det(Q^i{}_j)} \right) \tag{2.1}$$

with

$$E_{AB} = G_{AB} + B_{AB} \qquad \text{and} \qquad Q^i{}_j \equiv \delta^i{}_j + i\lambda \left[\Phi^i, \Phi^k\right] E_{kj}\,. \tag{2.2}$$

In writing (2.1) we have used a number of conventions taken from Myers [8]:

- Indices to be pulled-back to the worldline (see below) have been labelled by a. For other indices, the symbol A takes values in the full set of space-time coordinates while i labels only directions perpendicular to the center of mass world-line.

- The parameter λ is equal to l_s^2. While this convention differs by a factor of 2π from that of Myers [8], it will greatly simplify our presentation.

- The center of mass degrees of freedom decouple completely and are not relevant for our discussion. The fields Φ^i thus take values in the adjoint representation of SU(N). As a result, the fields satisfy $\mathrm{Tr}\,\Phi^i = 0$ and form a non-abelian generalization of the coordinates specifying the displacement of the branes from the center of mass. These coordinates have been normalized to have dimensions of $(length)^{-1}$ through multiplication by λ^{-1}.

The rest of the action is given by the non-abelian Chern-Simon terms. These involve the non-abelian 'pullback' P of various covariant tensors to the world-volume of the D0-brane e.g. $P(C^1_a) = C^1_a \frac{\partial x^a}{\partial t} = C^1_0 + \lambda C^1_i \frac{\partial \Phi^i}{\partial t}$, where we have used the static gauge $x^0 = t, x^i = \lambda\Phi^i$ for a coordinate x with origin at the D0-brane center of mass. The symbol STr will be used to denote a trace over the SU(N) index with a complete symmetrization over the non-abelian objects in each term. In this way, the Chern-Simons terms may be compactly written

$$S_{CS} = \mu_0 \int dt \, \mathrm{STr}\left(P\left[e^{i\lambda \hat{\imath}_\Phi \hat{\imath}_\Phi} (\sum C^{(n)} \, e^B)\right]\right) \,. \tag{2.3}$$

The symbol i_Φ is a non-abelian generalization of the interior product with the coordinates Φ^i,

$$i_\Phi \left(\frac{1}{2} C_{AB} dX^A dX^A \right) = \Phi^i C_{iB} dX^B \,. \tag{2.4}$$

A by now familiar property of this action is that it leads to the dielectric effect (or Myers effect) whereby a constant electric Ramond-Ramond 4-form field strength changes the classical ground state into a non-abelian solution known as the 'fuzzy two-sphere.' This solution is expected to represent a D2-brane made out of non-abelian D0-branes [8].

2.2 D0-branes in the D4 background

In this work wish to study the 'polarization' of the D0-brane bound state for the specific case of N D0-branes living in the space-time generated by a D4-brane. Such a background is defined by the metric G_{AB}, the dilaton ϕ, and the Ramond-Ramond 6-form field strength $F_{A_1 A_2 A_3 A_4 A_5 A_6}$:

$$\begin{aligned} ds_4^2 &= \mathcal{H}_4^{-1/2} \eta_{\mu\nu} dX^\mu dX^\nu + \mathcal{H}_4^{1/2} \delta_{mn} dX^m dX^n \\ e^{-2\phi} &= \mathcal{H}_4^{1/2} \\ F_{01234m} &= \partial_m \mathcal{H}_4^{-1} \,, \end{aligned} \tag{2.5}$$

with all other independent components of the field strength vanishing. Here the space-time coordinates described by the index A have been partitioned into directions parallel to the D4-brane (which we will label with a Greek index μ) and directions perpendicular to the D4-brane (which we label with a Latin index m). The function $\mathcal{H}_4 = 1 + \left(\frac{r_4}{|X|}\right)^3$ is the usual harmonic function of the D4-brane solution with $|X|^2 = \delta_{mn} X^m X^n$ and with $r_4 = (gN_4)^{1/3} l_s$ being the constant that sets the length scale of the supergravity solution.

To expand the Born-Infeld action in this background we first evaluate the pull-back $\frac{\partial x^a}{\partial t}(E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb})\frac{\partial x^b}{\partial t}$. Since $E_{it} = 0$, the only term not involving derivatives of Φ^i is $g_{tt} = \frac{\partial x^0}{\partial t} E_{tt} \frac{\partial x^0}{\partial t}$. For the terms that do involve Φ^i, the E_{ab} term exactly cancels the term $-g_{ai}\delta^{ij}g_{jb}$. Thus, since $B_{AB}=0$ we have only

$$\begin{aligned}\left(-P\left[E+E(Q^{-1}-\delta)E\right]\right)^{1/2} &= \left(-g_{tt}-\lambda^2\partial_t\Phi^i\partial_t\Phi^j Q^{-1}_{ij}\right)^{1/2}\\ &= \sqrt{-g_{tt}}\Bigg(1+\frac{\lambda^2}{2}(g_{tt})^{-1}\partial_t\Phi^i\partial_t\Phi^j g_{ij}+\frac{-i\lambda^3}{2}(g_{tt})^{-1}\times\\ &\qquad\times\partial_t\Phi^i\partial_t\Phi^j[\Phi^k,\Phi^l]g_{ik}g_{jl}+O(\lambda^4)\Bigg)\,.\end{aligned}$$

The factor $\sqrt{\det(Q^i_j)}$ generates the well known potential $[\Phi,\Phi]^2=[\Phi^i,\Phi^j]\times[\Phi^k,\Phi^l]g_{li}g_{jk}$ as well as higher order terms; i.e.

$$\sqrt{\det(Q)}=\left(1+\frac{\lambda^2}{4}[\Phi,\Phi]^2+\frac{i\lambda^3}{3!}[\Phi,\Phi]^3+O(\lambda^4)\right)\,. \tag{2.6}$$

Since the D4-brane background has $e^{-\phi}\sqrt{-g_{tt}}=1$, we find

$$\begin{aligned}S_{BI}=-T_0\int dt\ \mathrm{STr}\Bigg\{&1+\lambda^2\left(\frac{1}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j g_{ij}+\frac{1}{4}[\Phi,\Phi]^2\right)-\\ &-\lambda^3\left(\frac{i}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j[\Phi^k,\Phi^l]g_{il}g_{jk}+\frac{i}{3!}[\Phi,\Phi]^3\right)+O(\lambda^4)\Bigg\}.\end{aligned} \tag{2.7}$$

Finally, the dependence of the fields g_{AB} on the non-abelian scalars Φ^i must be determined through the Taylor expansions

$$g_{tt}=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\Phi^{i_1}\cdots\Phi^{i_n}\partial_{i_1\cdots i_n}g_{tt}\,,\qquad \text{and}\qquad g_{ij}=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\Phi^{k_1}\cdots\Phi^{k_n}\partial_{k_1\cdots k_n}g_{ij}\,. \tag{2.8}$$

This yields the following form for the Born-Infeld action, where from now on the symbols $g_{ij},\partial_k g_{ij},g_{tt},\partial_k g_{tt}$, *etc.* refer to the values of the fields at the D0-brane center of mass.

$$\begin{aligned}S_{BI}=-T_0\int dt\ \mathrm{STr}\Bigg\{&1+\lambda^2\left(\frac{1}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j g_{ij}+\frac{1}{4}[\Phi,\Phi]^2\right)+\\ &+\lambda^3\left(\frac{1}{2}\partial_t\Phi^i\partial_t\Phi^j\Phi^k\partial_k(g_{tt})^{-1}+\frac{1}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j\Phi^k\partial_k g_{ij}+\frac{1}{2}[\Phi,\Phi^i][\Phi^j,\Phi]\Phi^k\partial_k g_{ij}\right)-\\ &-\lambda^3\left(\frac{i}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j[\Phi^k,\Phi^l]g_{il}g_{jk}+\frac{i}{3!}[\Phi,\Phi]^3\right)+O(\lambda^4)\Bigg\}\,.\end{aligned} \tag{2.9}$$

The STr prescription removes the final two terms, which have been displayed separately on the third line. A careful study shows that the symmetrized trace in fact removes any term with an odd number of commutators $[\Phi,\Phi]$.

The only non-zero Chern-Simons terms involve the RR five form $C^{(5)}$. Hence, in direct analogue with Myers [8] we find the dipole coupling

$$S_{CS}=\mu_0\int d\tau\ \mathrm{STr}\left\{\frac{\lambda^3}{10}\Phi^i\Phi^j\Phi^k\Phi^l\Phi^m F_{0ijklm}+O(\lambda^4)\right\}\,. \tag{2.10}$$

Now the above approximations for the Born-Infeld and Chern-Simons actions are valid as long as the suppressed terms are of no significance. In the case of the Chern-Simons action the $O(\lambda^{3+k})$ term involves $4+k$ factors of Φ and k derivatives of F_{0ijklm}. Thus, the $O(\lambda^4)$ term can be neglected for $F \gg \lambda\Phi^i\partial F$, and similarly for the higher terms. Since the D4-brane background (2.5) is weak when the D4-brane is far away as we require, one expects $\lambda\Phi^i \sim \ell_s$. For such Φ, the $O(\lambda^4)$ terms are in fact small $|X| \sim r_4$ or greater. Similarly, this condition allows us to discard $O(\lambda^4)$ terms in the Born-Infeld action. Note that a study of the strong field effects on D0-branes near the D4-brane would require an understanding of the full non-abelian action to all orders. It is for this reason that we consider only distant D4-branes.

Imposing the above condition and assuming that the commutators are small, we arrive at the following (bosonic) effective action:

$$\begin{aligned} S_{\text{eff.}} = -T_0\lambda^2 \int dt\ \mathrm{STr}\Big\{ & \frac{1}{2g_{tt}}\partial_t\Phi\partial_t\Phi + \frac{1}{4}[\Phi,\Phi]^2 + \\ & + \lambda\bigg(\frac{1}{2}\partial_t\Phi^i\partial_t\Phi^j\Phi^k\partial_k(g_{tt})^{-1} + \frac{1}{2g_{tt}}\partial_t\Phi^i\partial_t\Phi^j\Phi^k\partial_k g_{ij} + \\ & + \frac{1}{2}[\Phi,\Phi^i][\Phi^j,\Phi]\Phi^k\partial_k g_{ij} + \frac{1}{10}\Phi^i\Phi^j\Phi^k\Phi^l\Phi^m F_{\tau ijklm}\bigg)\Big\}. \end{aligned} \tag{2.11}$$

These couplings also appear in [7, 12], along with the appropriate fermion terms. The fermion terms are rather long, and little insight is gained by writing them explicitly here. Drawing from [7, 12], the fermion terms will be introduced as needed below.

2.3 Deformations of the bound state

One might begin with a discussion of classical solutions corresponding to the above effective action. However, aside from the trivial commutative solution, one does not expect to find any static solutions.[1] We quickly note that, as opposed to the situation in [8], the BPS character of the commutative ground state forbids any non-abelian classical solutions from having lower energy. Furthermore, any classical dielectric effect in this context would amount to the formation of a D4/anti-D4 pair out of the D0-branes. A D4-brane by itself would remain static in the D4-background, while an anti-D4 brane would fall toward the background D4-brane. One therefore expects any bound state of D4 and anti-D4 branes to fall as well. Indeed, while one can find a static non-abelian solution (albeit an unstable one) for (2.11), the corresponding values of Φ are larger than the domain of validity of the expansion (2.11). One expects such a solution to be eliminated by a more complete treatment.

Nevertheless, we may expect that the non-abelian couplings to the background affect the quantum bound state by altering the shape of the potential and thus

[1] The literature [14, 15, 16] contains some examples of classical non-commutative solutions in similar (but non-supersymmetric) systems.

the ground state wavefunction. Let us calculate the size of the ground state by considering the expectation value of the squared radius operator $R^2 \equiv \lambda^2\,\mathrm{Tr}(\Phi^2) = \lambda^2\,\mathrm{Tr}(\Phi^i\Phi^j g_{ij})$. Note that by passing to an orthonormal frame one finds a full SO(9) spherical symmetry in the $O(\lambda^2)$ terms in our action. Thus, to $O(\lambda^2)$ the expectation value of $\mathrm{Tr}(\Phi^i\Phi^i g_{ii})$ is independent of i and R^2 is the radius of the corresponding sphere measured in terms of string metric proper distance.

Here we give a simple argument for the behavior of $\langle R^2\rangle$ based on the usual 't Hooft scaling behavior. There are, however, several subtleties that are pointed out below. A more complete discussion in terms of hamiltonian perturbation theory is given in the appendix A.

Our strategy is to treat the couplings to the D4-brane fields as perturbations to the D0-brane action in flat empty spacetime. Thus, we divide (2.11) into an 'unperturbed action' S_0 and a perturbation S_1. Note that as we place the N D0-branes far from the D4-branes, the Ramond-Ramond coupling term can be written in the form

$$\Phi^{\mu_1}\Phi^{\mu_2}\Phi^{\mu_3}\Phi^{\mu_4}\Phi^m F_{0\mu_1\mu_2\mu_3\mu_4 m} = \frac{f}{\lambda^{1/2}}\Phi^{\mu_1}\Phi^{\mu_2}\Phi^{\mu_3}\Phi^{\mu_4}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\Phi^m\frac{X^m}{|X|}\,, \tag{2.12}$$

where $f = 3(\frac{r_4^3\lambda^{1/2}}{z_\perp^4}) \approx 3\mathcal{H}_4^{-2}(\frac{r_4^3\lambda^{1/2}}{z_\perp^4})$ is a scalar dimensionless measure of the field strength. Here $z_\perp$ is the distance between the N D0-branes and the D4-brane, and $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ is the antisymmetric symbol on four indices. In what follows we treat all effects of the D4-brane only to lowest order in $(\mathcal{H}_4 - 1)$ and $f = 3(\frac{r_4^3\lambda^{1/2}}{z_\perp^4})$, so that $f \approx \mathcal{H}_4^{-2} f$. With this understanding, the other $O(\lambda^3)$ terms are also proportional to f.

It will be useful to express the dynamics in terms of rescaled fields and a rescaled time coordinate:

$$\tilde\Phi^i = \lambda^{1/2}\mathcal{H}_4^{1/12}(gN)^{-1/3}\Phi^i\,,\quad \tilde\Theta = \lambda^{3/4}\mathcal{H}_4^{1/8}(gN)^{-1/2}\Theta\,,\quad \tilde t = \lambda^{-1/2}\mathcal{H}_4^{-1/3}(gN)^{1/3}t\,., \tag{2.13}$$

where we have included the fermionic fields Θ for completeness. This yields the action [13]

$$S_0 = -N\int d\tilde t\ \mathrm{STr}\left(-\frac{1}{2}\partial_{\tilde t}\tilde\Phi\partial_{\tilde t}\tilde\Phi + \frac{1}{4}[\tilde\Phi,\tilde\Phi]^2 + \frac{1}{2}\tilde\Theta\dot{\tilde\Theta} - \frac{1}{2}\tilde\Theta\gamma^i[\tilde\Phi_i,\tilde\Theta]\right) \tag{2.14}$$

and the perturbation

$$\begin{aligned} S_1 = &-\left[(gN)^{1/3}\mathcal{H}_4^{-1/12}f\right]N\,\times \\ &\times\int d\tilde t\ \mathrm{STr}\Bigg(\frac{1}{10}\tilde\Phi^i\tilde\Phi^j\tilde\Phi^k\tilde\Phi^l\tilde\Phi^m\epsilon_{ijklm} + \frac{1}{2}\partial_t\tilde\Phi^i\partial_t\tilde\Phi^j\tilde\Phi^k\frac{\partial_k(g_{tt})^{-1}}{f} + \\ &\qquad + \frac{1}{2}\partial_t\tilde\Phi^i\partial_t\tilde\Phi^j\tilde\Phi^k\frac{\partial_k g_{ij}}{f} + \frac{1}{2}[\tilde\Phi,\tilde\Phi^i][\tilde\Phi^j,\tilde\Phi]\tilde\Phi^k\frac{\partial_k g_{ij}}{f} + \text{Fermions}\Bigg)\,, \\ \equiv &-[(gN)^{1/3}\mathcal{H}_4^{-1/12}f]\,\tilde S_1\,. \end{aligned} \tag{2.15}$$

Note that in writing $\tilde{S}_1$ we have extracted a factor of f from S_1. The advantage of this form is that both S_0 and $\tilde{S}_1$ are manifestly independent of g, λ, and f while they depend on N only through the overall factor and the trace. The dependence of S_0 and $\tilde{S}_1$ on $\mathcal{H}_4$ is only though contractions with g_{ij}. These could be further eliminated by passing to an orthonormal frame, so the dynamics of scalar contractions such as $\Phi^i\Phi^j g_{ij}$ will be independent of $\mathcal{H}_4$.

While we will not need the explicit form of the fermion terms in $\tilde{S}_1$, we will use the fact that each fermion term can be obtained from the bosonic terms by replacing three bosons with two fermions and an odd number of γ^i matrices.[2] This important property may be checked from the results of [12] and is related to the structure of the associated superfields. Note that the same relation holds between the purely bosonic term and the fermion term in the potential for S_0.

As a result of this structure, the fermion terms in $\tilde{S}_1$ again contain no explicit factors of g, λ, or f and have the same minimal dependence on $\mathcal{H}_4$ and N as the purely bosonic terms. It also follows that S_1 is antisymmetric under a total inversion of space in which the bosons Φ^i are mapped to $-\Phi^i$ and the fermions are rotated by π, effectively mapping the γ^i matrices to $-\gamma^i$.

Let us now consider the case $f = 0$ and the corresponding ground state $\langle R^2\rangle_0$. We will think of this as the limit of small $\ell_s/z_\perp$, so that we preserve $\mathcal{H}_4 \neq 1$. Note that we have

$$\langle R^2\rangle_0 = (gN)^{2/3}\mathcal{H}_4^{-1/6}\lambda\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle_0\,. \tag{2.16}$$

The factor $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle_0$ is manifestly independent of g, $\mathcal{H}_4$, and λ, and the form of S_0 is the usual one associated with 't Hooft scaling for which $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle_0$ is also independent of N in the limit of large N with gN fixed. This reproduces the results of [9, 1]:

$$\sqrt{\langle R^2\rangle_0} \sim (gN)^{1/3}\mathcal{H}_4^{-1/12}\lambda^{1/2}\,, \tag{2.17}$$

where the product of $(g\mathcal{H}_4^{-1/4})^{1/3}$ can be viewed as the natural dependence on the local string coupling $g_{\mathrm{local}} \equiv ge^{\phi} = g\mathcal{H}_4^{-1/4}$ of the D4-brane background.

Of course, the usual 't Hooft scaling argument is stated in terms of field theoretic perturbation theory and scattering states. As a result, there may be some subtlety in applying it to 0+1 systems. From the field theoretic viewpoint, such subtleties are associated with the infrared divergences typical of 0+1 dimensions. Nevertheless, we may take courage from the fact that (2.17) agrees with the upper bound in [9]. We will comment further on this below and the reader can find a more thorough treatment in the appendix A.

Let us now turn to the perturbed system. Considering the ground state expectation value as the low temperature limit of a thermal expectation value gives a euclidean path integral for $\langle R^2\rangle$. We wish to expand the factor e^{-S_1} $=$

[2] Essentially the number of such γ^i matrices is three, though sometimes pairs $\gamma^i\gamma^i$ appear contracted together and so form a matrix proportional to the identity that is not explicitly displayed.

$e^{-((gN)^{1/3}\mathcal{H}_4^{-1/12}f)\tilde{S}_1}$ as $1-(gN)^{1/3}\mathcal{H}_4^{-1/12}f\tilde{S}_1+(gN)^{2/3}\mathcal{H}_4^{-1/6}f^2\tilde{S}_1^2-\cdots$. Note that the order zero term gives just $\langle R^2\rangle_0$, the expectation value in the unperturbed ground state. The contribution from the first order term then vanishes because $\tilde{S}_1$ is anti-symmetric under a total inversion of space while R^2, S_0, and the integration measure are invariant.

Thus, the leading contribution is of second order in $\tilde{S}_1$ and we have

$$\frac{\langle R^2\rangle-\langle R^2\rangle_0}{\langle R^2\rangle_0}\approx(gN)^{2/3}\mathcal{H}_4^{-1/6}f^2\frac{\langle T\left[(\mathrm{Tr}\,\tilde{\Phi}^2)(\tilde{S}_1)^2\right]\rangle_0}{\langle\mathrm{Tr}\,\tilde{\Phi}^2\rangle_0}\,, \tag{2.18}$$

where the T represents time ordering. Again the factor $\langle T\left[(\mathrm{Tr}\,\tilde{\Phi}^2)(\tilde{S}_1)^2\right]\rangle_0$ is explicitly independent of g, λ, $\mathcal{H}_4$, and f. Furthermore, when written in terms of $\tilde{\Phi}^i$ and $\tilde{\Theta}^i$, $\tilde{S}_1$ and S_0 have the typical form associated with 't Hooft scaling for which $\langle\mathrm{Tr}\,\tilde{\Phi}^2\rangle$ should be independent of N in a system with action $S_0+f\tilde{S}_1$ in the limit of large N with gN fixed. Thus, we may express our final result as

$$\frac{\langle R^2\rangle-\langle R^2\rangle_0}{\langle R^2\rangle_0}\sim(gN)^{2/3}\mathcal{H}_4^{-1/6}f^2\,. \tag{2.19}$$

Here again one may ask about subtleties of the 't Hooft limit and infrared divergences in 0+1 dimensions. In particular, the factors of $\tilde{S}_1$ in (2.18) contain an integration over $\tilde{t}$. These integrals almost certainly diverge due to the fact that correlations do not die off at large times in 0+1 dimensional systems. To address such concerns, a more complete argument in terms of hamiltonian perturbation theory is provided in the appendix A. This consists of regulating the infrared divergence and working through the 't Hooft power counting for our case. If, in analogy with the usual 't Hooft argument at strong coupling, one takes as input that one should be able to read off certain properties of the full expectation values from an asymptotic expansion, then the arguments in the appendix A can be said to give a proof of (2.19).

3. The D0-D4 system in supergravity

Having considered the quantum mechanical description of the non-abelian D0-brane bound state, we now wish to compute a corresponding effect in classical supergravity. We seek a BPS solution containing both D0's and D4's with the D0's being both fully localized and separated from the D4-branes. It is conceptually simplest to discuss the full D0/D4 solution and then take a suitable decoupling limit. Such full solutions are known exactly, but only as an infinite sum over Fourier modes [19]. As a result, we find it more profitable here to follow a perturbative method as suggested by the quantum mechanical calculation above. We therefore expand the supergravity solution in f, the magnitude of the Ramond-Ramond 4-form field strength at the location of the zero-branes.

3.1 The perturbed solution

Let us consider a BPS system of D4-branes and N D0-branes with asymptotically flat boundary conditions. Using the usual isotropic ansatz in the appropriate gauge reduces the problem to solving the equations [20, 21, 22, 23]

$$\begin{aligned}\left(\partial_\perp^2 + \mathcal{H}_4 \partial_{||}^2\right) \mathcal{H}_0 &= 0\,, \\ \mathcal{H}_4 &= 1 + \left(\frac{r_4}{|X|}\right)^3\,,\end{aligned} \tag{3.1}$$

where as before the D4-brane lies at $X^m = 0$, $|X|^2 = \delta_{mn} X^m X^n$, and $\mathcal{H}_4$ and $\mathcal{H}_0$ are the 'harmonic' functions for the D4-brane and D0-brane respectively. The two relevant derivative operators are a flat-space laplacian ($\partial_{||}^2 \equiv \sum_{\mu=1}^{\mu=4} \partial_\mu \partial_\mu$) associated with the directions parallel to the D4-brane and another ($\partial_\perp^2 \equiv \sum_{m=5}^{m=9} \partial_m \partial_m$) associated with the perpendicular directions.

In order to treat the D4-branes as a perturbation, we place them far away from the D0-branes. It is convenient to change to new coordinates x^m (lowercase) whose origin is located at the D0 singularity. One of these coordinates is distinguished by running along the line connecting the D0- and D4-branes. Let us call this coordinate $x_\perp$. The other four x^m coordinates will play a much lesser role. Introducing the distance $z_\perp$ between the D0- and D4-branes and expanding $\mathcal{H}_4$ to first order about the new origin yields

$$\mathcal{H}_4 \approx \mathcal{H}_4(x=0) - 3\left(\frac{r_4}{z_\perp}\right)^3 \left(\frac{x_\perp}{z_\perp}\right) \equiv \mathcal{H}_4(0) + \delta\mathcal{H}_4\,. \tag{3.2}$$

Here we have used $z_\perp \gg (r_4, x_\perp)$, since the D4-branes are located far away. Note that fixing $z_\perp$ sets the location of the D0 singularity relative to the D4-brane. This is much like the fixing of the D0 center of mass degrees of freedom in section 2 as both set the overall location of the D0 branes. However, as we will see, it is not clear in general that the position of the singularity corresponds precisely to the center of mass. Equation 3.1 can be solved by expanding $\mathcal{H}_0$ in terms of $\delta\mathcal{H}_4$ i.e. $\mathcal{H}_0 = \mathcal{H}_{00} + \mathcal{H}_{01} + \cdots$ where $\mathcal{H}_{0n} = O(\delta{\mathcal{H}_4}^n)$. We find

$$\begin{aligned}(\partial_\perp^2 + \mathcal{H}_4(0)\partial_{||}^2)\mathcal{H}_{00} &= 0\,, \text{ so that } \mathcal{H}_{00} = 1 + \left(\frac{r_0}{r}\right)^7, \\ (\partial_\perp^2 + \mathcal{H}_4(0)\partial_{||}^2)\mathcal{H}_{01} &= \delta\mathcal{H}_4 \partial_{||}^2 \mathcal{H}_{00} \text{ and} \\ (\partial_\perp^2 + \mathcal{H}_4(0)\partial_{||}^2)\mathcal{H}_{02} &= \delta\mathcal{H}_4 \partial_{||}^2 \mathcal{H}_{01}\,.\end{aligned} \tag{3.3}$$

Here we have introduced $r^2 = |x|^2 \equiv \delta_{mn} x^m x^n + \mathcal{H}_4^{-1}(0)\delta_{\mu\nu} x^\mu x^\nu$, a sort of coordinate distance from the D0-brane. Note that this r does in fact label spheres of symmetry for the unperturbed solution $\mathcal{H}_{00}$.

Since the D4-brane background has altered the background metric for the D0-brane system, this will change certain familiar normalizations. We therefore note that total electric flux from the D0-branes must equal the number N of D0-brane charge quanta $g\ell_s^7/60\pi^3$. Since the D4-brane is far away, we may compute this flux in a region where $\mathcal{H}_0 \approx 1$ but where $\mathcal{H}_4 = \mathcal{H}_4(0)$. The result yields $gN\ell_s^7/60\pi^3 = \mathcal{H}_4^2 r_0^7$.

The above equations (3.3) are easily solved in terms of Green's functions. To first order the solution is:

$$\mathcal{H}_{01} = 7r_0^7\lambda^{-1/2} f \int dy^9 \frac{1}{\mid x-y \mid^7} \frac{y_\perp(9y_{\|}^2 - y^2)}{y^{11}}\,, \tag{3.4}$$

where $f = \frac{3\lambda^{1/2} r_4^3}{z_\perp^4}$. Here we have used $\mathcal{H}_4 \approx 1$ since (3.4) is already proportional to f.

While the integral is difficult to compute explicitly, the important features of $\mathcal{H}_{01}$ can be readily deduced. Note for example that under a rescaling of coordinates $y, x \to \beta y, \beta x$ the function $\mathcal{H}_{01}$ scales homogeneously as β^{-6}. As a result, we may write

$$\mathcal{H}_{01} = \frac{\omega_1 f}{x^6} r_0^7 \lambda^{-1/2}\,, \tag{3.5}$$

where ω_1 is an unknown dimensionless function of the angles associated with the direction cosines x^A/r. Furthermore, $\mathcal{H}_{01}$ is even under any $x^\mu \to x^\mu$ and under any $x^m \to x^m$ for $x^m \neq x_\perp$. However, we see that $\mathcal{H}_{01}$ is odd under $x_\perp \to -x_\perp$.

It is worthwhile to pause and understand the meaning of this anti-symmetry. One consequence is that first order effects of $\mathcal{H}_{01}$ on the spacetime curvature at the original bubble wall for a point with $x_\perp > 0$ are clearly opposite to those at the corresponding point with $x_\perp < 0$. Thus, if the curvature increases in one place it must decrease in the other. Since the perturbed wall is by definition the place where the metric has structure at the string scale, we see that if the wall moves toward the origin for $x_\perp > 0$ then it must move *away* from the origin for $x_\perp < 0$. Thus, the $\mathcal{H}_{01}$ term provides an (angle dependent) *shift* of the bubble so that it is no longer centered on the D0-brane center of mass. In particular, this has no effect on the *size* of the bubble.

We will return to this point below, but as a consequence we will need to study the second order term $\mathcal{H}_{02}$. We find

$$\mathcal{H}_{02} = 49r_0^7\lambda^{-1} f^2 \int dy^9 \frac{\omega_1}{\mid x-y \mid^7} \frac{y_\perp^2(9y_{\|}^2 - y^2)}{y^{11}}\,. \tag{3.6}$$

Again, scaling arguments show that we have

$$\mathcal{H}_{02} = \frac{\omega_2 f^2}{x^5}\,, \tag{3.7}$$

where ω_2 is again a dimensionless function of the angles. This time, however, $\mathcal{H}_{02}$ is even under $x^A \to x^A$ for any A. As a result, $\mathcal{H}_{02}$ directly encodes a change in the size

JHEP08(2001)043

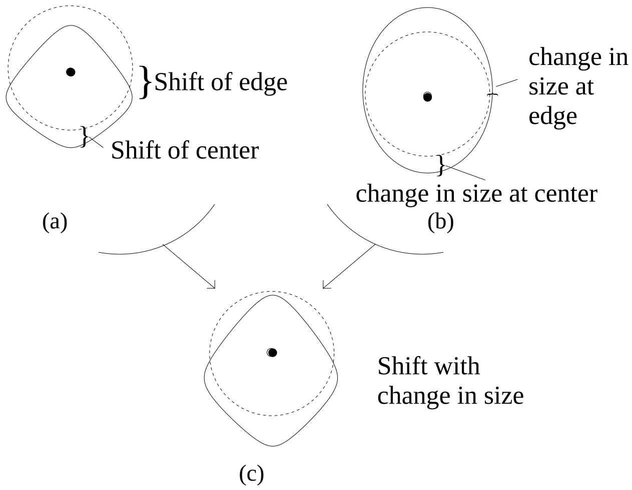

Figure 1: Two types of corrections to the radius of the N D0-brane system, first the shift of the boundary and second the change in size. Note that both effects are angle dependent. The unperturbed bubble wall has been shown as a dashed circle.

of the bubble. Although we have not evaluated the integrals (3.4), (3.6) explicitly, both expressions converge and could be computed numerically.

3.2 Analyzing the deformations

Let us now calculate the size of this solution. One might try to measure the size of the D0-brane configuration by studying properties of the Fourier transform (as in [18, 19]) or by identifying where the potential becomes of order one. However, it is clear that we wish to follow Polchinski [9] and use the measure that successfully reproduces the size of the unperturbed D0-brane bound state. While [9] described this correspondence in terms of quantities associated with the eleven-dimensional metric, we prefer to use the ten-dimensional string metric as this has been the setting for our discussion so far.

This means that we should locate the surface enclosing the D0-brane singularity where the string metric is so strongly curved that it has structure on the string scale. Inside this surface is a bubble of space that is well described by classical supergravity. However, when r is large the proper distance $2\pi \mathcal{H}_4^{1/4}\mathcal{H}_0^{1/4} r$ around the sphere enclosing the origin may still be on the order of ℓ_s so that the metric clearly has structure on the string scale. In particular, one might think of strings encircling the bubble itself. The region inside this surface is to be associated with the quantum D0-bound state, and one expects the bubble of 'normal' space and the bound state to have corresponding sizes and shapes.

Now, given the correspondence between the r of supergravity isotropic coordinates and the non-abelian Φ^2 in D0 quantum mechanics in the absence of the D4-brane, it is natural to expect a correspondence between the $R^2 = \mathrm{Tr}(\Phi^i \Phi^j g_{ij})$ of section 2 and the supergravity quantity $R^2 = \mathcal{H}_4^{1/2}\delta_{mn}x^m x^n + \mathcal{H}_4^{-1/2}\delta_{\mu\nu}x^\mu x^\nu = \mathcal{H}_4^{1/2} r^2$.

The edge of the bubble lies at the value of $R \equiv \mathcal{H}_4^{1/4} r$ for which $\ell_s \sim r\mathcal{H}_0^{1/4}\mathcal{H}_4^{1/4} = R\mathcal{H}_0^{1/4}(r)$. This yields the relation

$$\begin{aligned} R &\sim \ell_s[\mathcal{H}_0(r)]^{-1/4} \\ &\sim \ell_s \mathcal{H}_{00}^{-1/4}\left(1 - \frac{\mathcal{H}_{01}}{4\mathcal{H}_{00}} - \frac{\mathcal{H}_{02}}{4\mathcal{H}_{00}} + \frac{5}{20}\left(\frac{\mathcal{H}_{01}}{4\mathcal{H}_{00}}\right)^2 + O(f^3)\right). \end{aligned} \tag{3.8}$$

In order to make connections with the quantum mechanical calculations, we wish to consider this system in the decoupling limit $g \to 0$ with fixed gN and $r/\ell_s \sim g^{1/3} \to 0$. Note that this scaling may seem more familiar when expressed in terms of the eleven-dimensional plank mass $M_{11} = g^{-1/3}\ell_s^{-1}$ as it holds fixed the dimensionless quantity rM_{11}. We see that the corresponding asymptotic behavior of $\mathcal{H}_{00}$ is given by

$$\mathcal{H}_{00} \approx \left(\frac{r_0}{r}\right)^7 = \frac{gN\ell_s^7}{\mathcal{H}_4^{1/4}R^7}. \tag{3.9}$$

As a result, keeping only the order zero term yields an unperturbed value R_0 of R given by $R_0 \sim (gN)^{-1/4}\ell_s^{-3/4}\mathcal{H}_4^{7/16}R_0^{7/4}$, or $R_0 \sim (gN)^{1/3}\ell_s\mathcal{H}_4^{-1/12}$ in agreement with (2.16).

Recall that the effect of the $\mathcal{H}_{01}$ term is to shift the bubble by an angle-dependent amount but not to change the size of the bubble. Due to the angle-dependence, concepts like the radius R of the bubble also become angle-dependent. However, it is the average $\langle R^2\rangle$ of R^2 over the bubble that we expect to compare with expectation values in the quantum mechanical ground state. Taking such an average, it is clear that the term in (3.8) that is linear in $\mathcal{H}_{01}$ has no effect on $\langle R^2\rangle$. Of course, a shift of the bubble away from the origin will contribute to $\langle R^2\rangle$ at second order, and this effect is governed by the term $\frac{5}{20}\left(\frac{\mathcal{H}_{01}}{4\mathcal{H}_{00}}\right)^2$. This is in agreement with our quantum mechanical calculation where the effect on $\langle R^2\rangle$ appeared only at second order in f.

Averaging (3.8) over the bubble removes the linear term in $\mathcal{H}_{01}$, but the second order terms continue to contribute. Note that they both are of the form

$$\frac{\mathcal{H}_{02}}{4\mathcal{H}_{00}} \sim \frac{5}{20}\left(\frac{\mathcal{H}_{01}}{4\mathcal{H}_{00}}\right)^2 \sim (\text{const})f^2\langle R\rangle^2. \tag{3.10}$$

After taking the decoupling limit we find

$$\langle R\rangle \sim \frac{\langle R\rangle^{7/4}}{(gN)^{1/4}\mathcal{H}_4^{1/16}\ell_s^{3/4}}\left(1 + (\text{const})f^2\langle R\rangle^2\right), \tag{3.11}$$

since to the necessary order of accuracy we have $\langle R\rangle^n = \langle R^n\rangle$. Solving for $\langle R\rangle$ to second order in f yields the relation

$$\langle R\rangle \sim (\text{const}_1)l_s(gN)^{1/3}\mathcal{H}_4^{-1/12} + (\text{const}_2)\frac{1}{l_s^2}[l_s(gN)^{1/3}\mathcal{H}_4^{-1/12}]^2 l_s(gN)^{1/3}\mathcal{H}_4^{-1/12}f^2, \tag{3.12}$$

or

$$\frac{\langle R\rangle^2 - R_0^2}{R_0^2} \sim f^2 \mathcal{H}_4^{-1/6} (gN)^{2/3}\,, \tag{3.13}$$

in agreement with the quantum mechanical results of (2.19).

Finally, we may go one step further and interpret (3.13) in terms of the (angle dependent) shift of the bubble away from the D0 singularity[3] ($shift$) and the change in the size ($\Delta size$), let us write $R = R_0 + shift + \Delta size$ where we think of each quantity as depending on the direction in which it is measured from the D0-brane singularity and where we take $shift$ to be proportional to f and $\Delta size$ to be proportional to f^2 as indicated by our discussion above. Averaging over all directions we find

$$\langle (R_0 + shift + \Delta size)^2\rangle - R_0^2 = R_0 \langle shift\rangle + R_0 \langle \Delta size\rangle + \langle (shift)^2\rangle + O(f^3)\,, \tag{3.14}$$

where of course, $\langle shift\rangle = 0$ to lowest order[4].

Since both second order contributions to (3.8) were of the same order, we have

$$\begin{aligned} \Delta size &\sim R_0 f^2 (gN)^{2/3} \mathcal{H}_4^{-1/6}\,, \qquad \text{and}\,, \\ shift &\sim R_0 f (gN)^{1/3} \mathcal{H}_4^{-1/12}\,. \end{aligned} \tag{3.15}$$

This is more information than was obtained from the quantum mechanical calculations as in that case we were unable to separate the contributions of the shift and change in size. Nevertheless, (3.15) is consistent with the quantum mechanical results.

4. Discussion

In the above work we have analyzed deformations of the non-abelian D0-brane bound state and of the bubble of 'normal space' in the near D0-brane supergravity solution. In each case, the deformations are polarizations induced by the presence of a distant D4-brane and the deformations were shown to have corresponding scaling properties. This supports the idea that the gravity/gauge theory duality associated with D0-branes can be extended to include couplings to nontrivial backgrounds such as those discussed in [12, 7, 8]. A part of this was the analysis in the appendix of infrared issues associated with 't Hooft scaling in 0+1 dimensions. This in turn strengthens the argument that Polchinski's upper bound [9] on the size of the D0-brane bound state in fact gives the full scaling with N.

[3] I.e., the shift of the center of the bubble away from the D0 singularity. Unfortunately, it is unclear how to identify the separation of the center and the singularity in terms of the gauge theory description of section 2. Note that this is not the same as the center of mass U(1) degrees of freedom, which specify the overall location of the-D0 branes relative to the D4-brane.

[4] The contribution to $\langle shift\rangle$ from those points that are not moved across the $x_\perp = 0$ plane averages to zero, but the contribution from those points that are moved across this plane does not. However, the fraction of points moved across the plane is proportional to $shift/R_0$, so this simply generates an additional second order contribution of size $(shift)^2$.

One might expect our polarization effects to be particularly interesting when the D0- and D4-branes are close together and the effects are strong. Indeed, it is known that the D0-brane contribution to the supergravity solution is greatly distorted in the limit of zero separation [19]. However, pursuing such an analysis in D0-brane quantum mechanics would require an understanding of the full non-abelian dynamics as a truncation to $O(\lambda^3)$ or any other finite order would no longer be sufficient.

It is worth taking a moment to contrast the polarization effects studied here with those studied in [19] which were argued to be dual to certain effects in the D4-brane field theory. While both works considered a change in the characteristic size of the D0-brane supergravity solution induced by the D4-brane, the details are rather different. The previous work [19] used a measure of the D0-brane size in supergravity that followed from purely classical considerations. In particular, the curvature did not reach the string scale near the edge of the D0-brane region and the corresponding size could be expressed entirely in terms of supergravity quantities without using any explicit factors of ℓ_s. In contrast, in the present work we have followed Polchinski [9] and used a measure of size in which a supergravity result (the distance around the D0-brane bubble) is explicitly set equal to ℓ_s. It is therefore clear that these works use quite different measures of the classical 'size' of the D0-brane solution and discuss different physics.

While this is clear with hindsight, it is also somewhat surprising. Let us recall that the discussion of [19] compared the D0-branes in a supergravity solution with D0-branes as represented by solitons (Yang-Mills instantons) in the D4-brane field theory. The associated measure of size in supergravity was argued to correspond to the scale size of the instantons. One might well think it natural that the scale size of such instantons be connected to the size of a D0-bound state in matrix theory. We see, however, that the relation between matrix theory and instantons in the D4-brane theory must be more subtle. It would be of interest to understand this in detail.

Finally, let us consider extensions to other Dp/D$(p+4)$ systems. Having set up the framework, such calculations are straightforward. Adapting our covariant scaling argument to the p+1 dimensional unperturbed bosonic action

$$S_0 = \frac{\lambda^2}{g} \int d^{p+1}x \ \mathrm{STr}\left[e^{-\phi} \left(\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}[\Phi,\Phi]^2 \right) \right], \tag{4.1}$$

suggests the rescaling

$$\tilde{\Phi}^i = \lambda^{1/2}(gN\mathcal{H}_{p+4}^{-1/4})^{-1/(3-p)}\Phi^i , \qquad \tilde{x} = \lambda^{-1/2}(gN\mathcal{H}_{p+4}^{-1/4})^{1/(3-p)}x , \tag{4.2}$$

where $\mathcal{H}_{p+4}$ is the D$(p+4)$-brane harmonic function associated with the background in which the Dp-branes are to be placed. This yields

$$\frac{\langle R^2\rangle - \langle R^2\rangle_0}{\langle R^2\rangle_0} \sim (gN\mathcal{H}_4^{-1/4})^{2/(3-p)} f_{p+4}^2 , \tag{4.3}$$

where $f_{p+4} = \ell_s \partial_{x_\perp} \mathcal{H}_{p+4}$ is again a dimensionless measure of the field strength produced by the distant D$(p+4)$-brane.

The supergravity calculation is correspondingly straightforward. The only change is to modify a single power in (3.11) to yield

$$\langle R \rangle \sim \frac{\langle R \rangle^{(7-p)/4}}{(gN\mathcal{H}_{p+4}^{1/4})^{1/4}\ell_s^{3/4}} \left(1 + (\mathrm{const})\mathrm{f}_{\mathrm{p+4}}^2 \langle \mathrm{R} \rangle^2\right). \tag{4.4}$$

Solving this equation as before one finds

$$\begin{aligned} \langle R \rangle \sim & (\mathrm{const}_1) l_s (gN\mathcal{H}_{p+4}^{-1/4})^{1/(3-p)} + \\ & + (\mathrm{const}_2) \frac{1}{l_s^2} [l_s (gN\mathcal{H}_{p+4}^{-1/4})^{1/(3-p)}]^2 l_s (gN\mathcal{H}_{p+4}^{-1/4})^{1/(3-p)} f_{p+4}^2, \end{aligned} \tag{4.5}$$

or

$$\frac{\langle R \rangle^2 - R_0^2}{R_0^2} \sim f_{p+4}^2 (gN\mathcal{H}_{p+4}^{-1/4})^{2/(3-p)}. \tag{4.6}$$

Again, we find complete agreement between the quantum mechanical and supergravity calculations. However, for $p = 3$ the above 'solutions' diverge and there is in fact no solution of this form. Larger branes are troubled by the issues that the associated quantum field theories are not renormalizable and that there is in fact no regime in which the field of the $p + 4$ brane is weak. However, the results should be meaningful for the cases $p = -1$, $p = 1$ and $p = 2$. Regarding the issue of infrared divergences in the quantum mechanical argument, these should not be important for $p = 2$ and they are manifestly absent for $p = -1$ since the worldvolume of an instanton is compact. For $p = 1$ one assumes that a more careful argument, perhaps along the lines of that in the appendix A, should be able to deal with any infrared divergences. Thus our conclusions should extend to these cases as well.

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A. Hamiltonian treatment of the D0-brane quantum mechanics

In this appendix we provide a more complete argument for the scaling with f, g, λ, $\mathcal{H}_4$, and N of the deformations in the quantum mechanical bound state. The primary goal is to treat the infrared behavior with due care. This we succeed in doing, though certain technical assumptions will need to be introduced as discussed below. It is convenient to separate the discussion of the dependence on g, λ, $\mathcal{H}_4$, and f (which essentially translates the arguments of section 2 into hamiltonian language) from the discussion of the dependence on N. The latter will involve reconsidering the derivation of 't Hooft scaling while controlling the infrared behavior. This strengthens the argument that Polchinski's upper bound [9] on the size of the D0 bound state indeed gives the full scaling with N.

A.1 Scaling with g, λ, $\mathcal{H}_4$, and f

Since we work with a 0+1 system, the most control can be obtained in the hamiltonian formulation. As in section 2.3, it is useful to introduce rescaled bosonic fields

$$\tilde{\Phi}^i = \lambda^{1/2}\mathcal{H}_4^{1/12}(gN)^{-1/3}\Phi^i\,, \qquad \tilde{P}_i = \lambda^{-1/2}\mathcal{H}_4^{-1/12}(gN)^{1/3}P_i\,, \tag{A.1}$$

where P_i is the (matrix valued) canonical momentum conjugate to Φ^i and where $\tilde{P}_i$ is conjugate to $\tilde{\Phi}^i$. We also introduce the rescaled fermions $\tilde{\Theta} = \lambda^{3/4}\mathcal{H}_4^{1/8}(gN)^{-1/2}\Theta$ and observe from the normalization of the actions (2.11) and (2.14) that the rescaled fermions satisfy anti-commutation relations of the form $\{\tilde{\Theta},\tilde{\Theta}\}_+ \sim \frac{1}{N}$. In particular, these anti-commutation relations are independent of g, λ, $\mathcal{H}_4$, and f.

In order to establish the scaling with f, g, $\mathcal{H}_4$, and λ, it is useful to proceed much as in section 2 and write the hamiltonian H associated with the action (2.11) in form $H_0 + H_1$ for

$$\begin{aligned}
H_0 &\equiv \left[(gN)^{1/3}\mathcal{H}_4^{-1/12}/\lambda^{-1/2}\right]\tilde{H}_0\,,\\
&= \left[(gN)^{1/3}\mathcal{H}_4^{-1/12}/\lambda^{-1/2}\right] N\,\mathrm{STr}\left\{\frac{1}{2}(N^{-1}\tilde{P}^i)(N^{-1}\tilde{P}_i) + \frac{1}{4}[\tilde{\Phi},\tilde{\Phi}]^2 + \text{Fermions}\right\},\\
H_1 &\equiv \left[(gN)^{2/3}\mathcal{H}_4^{-1/6}f/\lambda^{-1/2}\right]\tilde{H}_1\,,\\
&= \left[(gN)^{2/3}\mathcal{H}_4^{-1/6}f/\lambda^{-1/2}\right] N\times\\
&\quad\times \mathrm{STr}\left\{\frac{1}{10}\tilde{\Phi}^a\tilde{\Phi}^b\tilde{\Phi}^c\tilde{\Phi}^d\tilde{\Phi}^e\epsilon_{abcde} + \frac{1}{2}\partial_t\tilde{\Phi}^i\partial_t\tilde{\Phi}^j g_{ij}\tilde{\Phi}^k\frac{\partial_k(g_{tt})^{-1}}{f} + \frac{1}{2}\partial_t\tilde{\Phi}^i\partial_t\tilde{\Phi}^j\tilde{\Phi}^k\frac{\partial_k g_{ij}}{f} + \right.\\
&\qquad\left. + \frac{1}{2}[\tilde{\Phi},\tilde{\Phi}^i][\tilde{\Phi}^j,\tilde{\Phi}]\tilde{\Phi}^k\frac{\partial_k g_{ij}}{f} + \text{Fermions} + \text{higher terms}\right\}.
\end{aligned} \tag{A.2}$$

Here $\epsilon_{abcde} = \pm 1, 0$ is the anti-symmetric symbol in the $x_\mu, x_\perp$ directions. Note that $\tilde{H}_0$ and $\tilde{H}_1$ are manifestly independent of f, g, and λ. They depend on $\mathcal{H}_4$ only through the appearance of the metric g_{ij} in contractions. As in section 2, the factors of g_{ij} would disappear if an orthonormal frame were introduced. The result is that the dependence on $\mathcal{H}_4$ through g_{ij} does not affect scalars such as R^2.

We now wish to calculate $\langle R^2\rangle$ by treating H_1 as a perturbation to H_0. Although the D0-brane bound state is normally thought of as degenerate, it is sufficient to use nondegenerate perturbation theory here. The point here is that the ground state degeneracy is associated with supersymmetry and that states in the corresponding multiplet are generated by the action of the fermionic zero modes associated with the center of mass. This follows from the observation that the ground states of the N D0-brane bound state must be in one-to-one correspondence with the ground states of a single D0-brane as required by the interpretation of the N D0-brane bound state as the eleven dimensional supergraviton with N units of momentum. Recall that we have discarded such center of mass degrees of freedom (whether Bosonic or fermionic) as they decouple from the dynamics of interest. Thus, the ground state of our H_0 is in fact non-degenerate.

As in section 2, the perturbation H_1 is odd under spatial inversion while H_0 and R^2 are even. As a result, the first order contribution to $\langle R^2 \rangle$ vanishes. The leading contribution thus comes from the second order term. We will make use of the operator $\bar{P}$, which can be expressed as the following sum over the eigenstates $|k\rangle_0$ of H_0:

$$\bar{P} = \sum_{k \neq 0} \frac{|k\rangle_{00}\langle k|}{E_k} \equiv (gN)^{-1/3} \mathcal{H}_4^{1/12} \lambda^{+1/2} \tilde{\bar{P}} \,. \tag{A.3}$$

Note that this definition makes $\tilde{\bar{P}}$ independent of g, λ, $\mathcal{H}_4$, and f. Since the ground state energy of H_0 vanishes, the second order contribution may be written

$$\begin{aligned} \langle R^2 \rangle - \langle R^2 \rangle_0 &= \langle\, R^2 \bar{P} H_1 \bar{P} H_1 + H_1 \bar{P} R^2 \bar{P} H_1 + H_1 \bar{P} H_1 \bar{P} R \,\rangle_0 \,, \\ &= [(gN)^{4/3} f^2 \lambda] \langle \left(\mathrm{Tr}\, \tilde{\Phi}^2 \right) \tilde{\bar{P}} \tilde{H}_1 \tilde{\bar{P}} \tilde{H}_1 + \tilde{H}_1 \tilde{\bar{P}} \left(\mathrm{Tr}\, \tilde{\Phi}^2 \right) \tilde{\bar{P}} \tilde{H}_1 + \\ &\qquad + \tilde{H}_1 \tilde{\bar{P}} \tilde{H}_1 \tilde{\bar{P}} \left(\mathrm{Tr}\, \tilde{\Phi}^2 \right) \rangle_0 . \end{aligned} \tag{A.4}$$

As a result, the scaling with g, f, $\mathcal{H}_4$, and λ is clearly

$$\frac{\langle R^2 \rangle - \langle R^2 \rangle_0}{\langle R^2 \rangle_0} \sim g^{2/3} \mathcal{H}_4^{-1/6} f^2 \,, \tag{A.5}$$

in accord with (2.19).

Based on the analogy with 't Hooft scaling, one expects that the dependence on N (in the 't Hooft limit) is also manifest in equation (A.4) and that (A.5) scales like $N^{2/3}$, again in agreement with (2.19). Since, however, this is less than obvious in the present setting, we now provide a separate argument for the dependence on N.

A.2 Scaling with N

This new argument is based on the usual 't Hooft power counting, though it will be done in the framework of hamiltonian perturbation theory and we will take due care with regard to the infrared behavior. Nevertheless, we will need to take as input that certain features of the expectation values can be read off from the (asymptotic) perturbation series in a straightforward manner. This assumption is in direct parallel with assumption made when 't Hooft scaling is applied to field theories (i.e., in more than 0+1 dimension) at strong coupling. These technical details will be discussed further below.

As in the familiar 't Hooft argument, the perturbation theory organizes itself nicely when one studies perturbations about the free (i.e., quadratic) hamiltonian. Thus, we will need to treat even the potential terms present in H_0 above as perturbations. Note that these terms are not small. For this reason as well as others associated directly with infrared effects as discussed below, we will not be able to truncate the perturbation expansion at any finite order. Instead, we need to consider the entire perturbation series. We will assume that this series can be summed in some sense and that this sum describes all of the physics of interest.

For the argument below we introduce $\tilde{f} = (gN)^{1/3}\mathcal{H}^{-1/12}f$ and consider the rescaled hamiltonian

$$\begin{aligned}\tilde{H} &= H_{free} + \delta H = [(gN)^{-1/3}\lambda^{1/2}\mathcal{H}^{1/12}]H \\ &= \mathrm{STr}\Bigg(\frac{1}{2}\frac{\tilde{P}^2}{N} + \frac{N}{4}[\tilde{\Phi},\tilde{\Phi}]^2 + \tilde{f}N\Big\{\frac{1}{10}\tilde{\Phi}^a\tilde{\Phi}^b\tilde{\Phi}^c\tilde{\Phi}^d\tilde{\Phi}^e\epsilon_{abcde} + \frac{1}{2}\partial_t\tilde{\Phi}^i\partial_t\tilde{\Phi}_j\tilde{\Phi}^k\frac{\partial_k(g_{tt})^{-1}}{f} + \\ &\qquad + \frac{1}{2}\partial_t\tilde{\Phi}^i\partial_t\tilde{\Phi}^j\tilde{\Phi}^k\frac{\partial_k g_{ij}}{f} + \frac{1}{2}[\tilde{\Phi},\tilde{\Phi}^i][\tilde{\Phi}^j,\tilde{\Phi}]\tilde{\Phi}^k\frac{\partial_k g_{ij}}{f}\Big\} + \text{Fermions}+ \\ &\qquad + \text{higher terms}\Bigg). \end{aligned} \tag{A.6}$$

Here H_{free} contains only the quadratic terms while δH contains all others.

The corresponding system clearly depends on f, g, and λ only through the combination $\tilde{f}$. The dependence on $\mathcal{H}_4$ is trivially removed as usual by passing to an orthonormal frame. The higher terms will in fact be relevant to our argument, though all that is important is that they may again be written with only a single explicit factor of N by absorbing appropriate factors of (gN) into the coupling constants as we have done in rewriting f in terms of $\tilde{f}$.

We wish to show that the dependence of $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle$ and similar expectation values also arises only through $\tilde{f}$ (and similar rescaled coupling constants). This will guarantee that (A.5) scales with $N^{2/3}$ as desired. Note that $\tilde{H}$ and H have the same ground state. As a result, expectation values like $\langle\tilde{\Phi}^2\rangle$ are identical for the system defined by H and the one defined by $\tilde{H}$.

If H_{free} is to have a normalizeable ground state, then the Φ^i degrees of freedom must be compactified. The simplest approach is to replace each component $(\Phi^i)^a_b$ in the matrix Φ^i with $L\sin(\phi^{ia}_b/L)$ for an angular variable ϕ^{ia}_b that lives on a circle and takes values in $[-\pi L, \pi L]$. In the limit of large L this reproduces the usual Φ^i fields which are valued on the real line.[5]

[5]Actually, it reproduces $2^{n_{boson}}$ decoupled copies of our system where $n_{boson} = 9(N^2-1)$ is the total number of such bosonic degrees of freedom. However, this will not effect the analysis in any

Having compactified the bosons, we may expand the expectation value $\langle \mathrm{Tr}\, \tilde{\Phi}^2 \rangle = \langle L^2 \sum_{a,b} g_{ij} \sin(\phi_b^{ia}/L) \sin(\phi_a^{ib}/L) \rangle$ perturbatively in the interaction δH. At order k, each term in this expression will involve the expectation value in the free ground state of $L^2 \sum_{a,b} \sin(\phi_b^{ia}/L) \sin(\phi_a^{ib}/L)$ times k factors of δH and P_{free} where

$$P_{\text{free}} = \sum_{n \neq 0} \frac{|n\rangle\langle n|}{E_n} \tag{A.7}$$

is a sum over eigenstates of H_{free} and n is a $9(N^2-1)$ dimensional vector giving the mode numbers of the associated momentum eigenstates for each bosonic degree of freedom. The states $|n\rangle$ are normalized and so have wavefunctions

$$\langle \phi | n \rangle = \frac{1}{(2\pi L)^{9(N^2-1)}} \exp(in\phi) \tag{A.8}$$

with associated H_{free} eigenvalues $E_n = \frac{n^2}{NL^2}$.

Since each term contains the same number of factors of P_{free} and δH, we may shift a factor of N from one to the other and write this series in terms of powers of $\frac{1}{N}\delta H$ and

$$NP_{free} = \frac{(NL)^2}{n^2} \sum_{n \neq 0} |n\rangle\langle n|. \tag{A.9}$$

The advantage of this organization is that $\frac{1}{N}\delta H$ has no explicit dependence on N while the explicit dependence of NP_{free} on N is only through the combination NL.

Each term of order k has k factors of NP_{free} and has therefore been written as $(NL)^{2k}$ times a sum over k mode vectors $n_1, \ldots, n_k$. The summand is the product of $n_1^{-2} n_2^{-2} \cdots n_k^{-2}$ multiplied both by one matrix element of the form

$$\langle n_{j_1} | L^2 \sum_{a,b} \sin(\phi_b^{ia}/L) \sin(\phi_a^{ib}/L) | n_{j_2} \rangle \tag{A.10}$$

and by k matrix elements of the form $\frac{1}{N}\langle n_{j_1} | \delta H | n_{j_2} \rangle$, so that there are $k+1$ matrix elements in all.

Each of the associated operators ($\mathrm{Tr}\, \tilde{\Phi}^2$ and δH) can be written as a sum of monomials in fermions and $L \sin(\phi_b^{ia}/L)$. It is simplest to first run through the argument ignoring the fermion terms, and then to address their effects separately in section A.3 below. Consider then a purely bosonic matrix element from the above product having α factors of $L \sin(\phi_b^{ia}/L)$. This matrix element takes the form

$$\sum_{a_1, \ldots, a_\alpha} L^\alpha \langle n_{j_1} | \sin(\phi_{a_2}^{i_1 a_1}/L) \sin(\phi_{a_3}^{i_2 a_2}/L) \cdots \sin(\phi_{a_1}^{i_\alpha a_\alpha}/L) | n_{j_2} \rangle A_{i_1, \ldots, i_\alpha}, \tag{A.11}$$

where $A_{i_1, \ldots, i_\alpha}$ is a tensor that does not depend on L or N.

way.

Let us begin by considering 'non-diagonal tensors' $A_{i_1,\ldots,i_\alpha}$ which vanish when any two indices coincide. The general tensor $A_{i_1,\ldots,i_\alpha}$ can be written as a sum of such non-diagonal tensors and 'purely diagonal' tensors (which vanish *unless* two or more indices coincide). It is useful to note that the states $|n\rangle$ can be written as the tensor product $\otimes_{i,a,b}|(n_j)^{ia}_b\rangle$, where each momentum eigenstate $|(n_j)^{ia}_b\rangle$ lives in a Hilbert space $L^2(\mathbf{R})$ associated with the single bosonic degree of freedom ϕ^{ia}_b. Using this observation and again assuming a non-diagonal tensor $A_{i_1,\ldots,i_\alpha}$, the matrix elements (A.11) factor as

$$\sum_{a_1,\ldots,a_\alpha} \bar{\delta}_{n_{j_1},n_{j_2}} L^\alpha \langle (n_{j_1})^{ia_1}_{a_2} | \sin(\phi^{i_1 a_1}_{a_2}/L) | (n_{j_2})^{ia_1}_{a_2}\rangle \times$$

$$\times \langle (n_{j_1})^{ia_2}_{a_3} | \sin(\phi^{i_2 a_2}_{a_3}/L) | (n_{j_2})^{ia_1}_{a_2}\rangle \cdots \times \tag{A.12}$$

$$\times \langle (n_{j_1})^{ia_\alpha}_{a_1} | \sin(\phi^{i_\alpha a_\alpha}_{a_1}/L) | (n_{j_2})^{ia_\alpha}_{a_1}\rangle A_{i_1,\ldots,i_\alpha}\,, \tag{A.13}$$

where $\bar{\delta}_{n_{j_1},n_{j_2}}$ is a Kronecker delta function that sets n_{j_1} equal to n_{j_2} except for those components of n_{j_1}, n_{j_2} that still appear explicitly in (A.13).

Note that each factor

$$\langle (n_{j_1})^{ja}_b | \sin(\phi^{ja}_b/L) | (n_{j_2})^{ja}_b\rangle = \frac{1}{2i}\left(\delta_{(n_{j_1})^{ja}_b,(n_{j_2})^{ja}_b+1} - \delta_{(n_{j_1})^{ja}_b,(n_{j_2})^{ja}_b-1}\right) \tag{A.14}$$

is independent of N or L while the sum over $a_1,\ldots,a_\alpha$ contributes N^α similar terms. The sum over spatial directions $i_1,\ldots i_\alpha$ generates a purely numerical term controlled by the tensor $A_{i_1,\ldots,i_\alpha}$ which is thus independent of N and L.

Consider now the full contribution from a term at order k which is built entirely from non-diagonal tensors. This yields a factor of $(NL)^{2k+\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}$ times a sum over $n_1,\ldots,n_k$. This final sum contains no explicit factors of N or L. Recall that components of n_{j_1}, n_{j_2} that do not appear explicitly in (A.13) are controlled by the Kronecker delta function $\bar{\delta}_{n_{j_1},n_{j_2}}$, while those appearing explicitly in (A.13) are controlled by the Kronecker delta functions in (A.14). Thus, the final sum contains at least one Kronecker delta function for each component of each n_i over which the sum is performed. As a result, the number of terms contributing to the final sum is independent of N or L (though it does depend on $\alpha_1,\alpha_2,\ldots\alpha_{k+1}$). We conclude that the full contribution from each term at order k is a purely numerical factor times $(NL)^{2k+\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}$. The important observation here is that the result depends only on the product NL.

Let us now consider a term with a purely diagonal tensor. While the bosonic fields ϕ^{ia}_b involved in a given matrix element might still be distinct, they need not necessarily be so. Thus there may be factors of $\langle (n_{j_1})^{ia}_b | L^m \sin^m(\phi^{ia}_b/L) | (n_{j_2})^{ia}_b\rangle$ for $m > 1$. Such matrix elements are again of order one times Kronecker deltas. However, there are only N such terms compared to the $\binom{N}{m}$ terms involving m different factors of the form $\langle (n_{j_1})^{ia}_b | L \sin(\phi^{ia}_b/L) | (n_{j_2})^{ia}_b\rangle$. Thus, at a given order in L, the contribution from terms with repeated bosonic fields in a given matrix element is suppressed by factors of N relative to terms with distinct bosonic fields in each matrix element. It follows that the contribution of purely diagonal tensors is again of the form $(NL)^{2k+\alpha_1+\alpha_2+\cdots+\alpha_k+\alpha_{k+1}}$ times a numerical factor plus sub-leading terms in N.

We should now take the large L limit and then take the limit of large N. Clearly, however, this limit fails to converge if we truncate the expansion at a fixed order k. From the field theory perspective, this is the usual infrared divergence of perturbation theory in low dimensional systems. Thus, we should think of using the perturbation series as a whole to extract information about the full function $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle$ of N, L. That is, the series must first be summed and then the large L limit may be taken. We have seen that the sum has one contribution from terms that involve only the product NL and another from terms that are sub-leading in N (at a fixed order in L). We now make the (reasonable but not necessarily justified) assumption that at large N we may neglect the terms that are sub-leading in N at a given order in L. Note that this assumption mirrors that of the field theory application of 't Hooft scaling at strong coupling. There, the 't Hooft scaling argument assumes that terms sub-leading in N at a given order in the 't Hooft coupling can be neglected even though one is not in the perturbative regime.

After discarding such sub-leading terms, What remains is a function only of the product NL. Thus, we expect that at large N the function $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle$ depends only on the product NL. Finally, the full hamiltonian should have a normalizeable ground state[6] even for infinite L so that the limit $L \to \infty$ of $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle$ is finite and thus independent of L. Since, however, this function depended on N only through the product NL, it follows immediately that the large L limit of $\langle \mathrm{Tr}\,\tilde{\Phi}^2\rangle$ is also independent of N. This is exactly the result we desired to show.

A.3 Fermions

Of course, we must still consider the effect of the fermion terms. Note that there is an important effect of the fermions already at the kinematic level, in that the fermion zero modes cause the free ground state to be massively degenerate. For our free hamiltonian, all fermion degrees of freedom contribute to such a degeneracy, not just those associated with the center of mass. We expect this degeneracy to be completely lifted, but we should ensure that we are indeed computing the expectation value in the correct ground state.

As is usual in degenerate perturbation theory, we use a basis of free ground states in which the perturbation δH is diagonal. This ensures that the perturbation series takes the same form as in (A.2), and that there is no danger of dividing by a vanishing

[6]This is the assumption that the presence of a distant D4-brane does not completely disrupt the D0 bound state and is to be expected on the basis of both supersymmetry arguments and supergravity physics. Note however that this assumption would be false if we applied it only to the terms explicitly displayed in (A.6) since the quintic term is unbounded below.

energy denominator E_n. Since this degeneracy should be completely lifted by the interaction, one particular state in this basis will lead to the true ground state. Let us call this state $|0\rangle$. In contrast, we will denote the original fermion vacuum $|vac\rangle$. Since δH is invariant under both SO(10) rotations and SU(N), both the true ground state and $|0\rangle$ must be as well.

Now, recall that any operator in the space of free ground states can be expressed as a finite polynomial in $\tilde{\Theta}^{ia}_b$. Consider in particular this representation of the projection operator $|0\rangle\langle 0|$. Invariance under SU(N) tells us that this operator may be expressed as a sum of the traces $\mathrm{Tr}(\tilde{\Theta}^{1_i} \cdots \tilde{\Theta}^{i_k})$. Thus, we may write

$$|0\rangle = \sum_\beta A^\beta_{i_1,\cdots,i_\beta} \tilde{\Theta}^{i_1} \ldots \tilde{\Theta}^{i_\beta} |vac\rangle \,, \tag{A.15}$$

for an appropriate collection of tensors $A^\beta_{i_1,\ldots,i_\beta}\tilde{\Theta}^{i_1}$. These tensors are clearly independent of L. To investigate their dependence on N, it is important to understand the normalization of $\tilde{\Theta}^i$ properly. From [12], we find (in terms of our conventions) that the original action S contains a factor of $\frac{\mathcal{H}_4^{1/4}\lambda^{3/2}}{g}$ in front of the fermion kinetic term. Thus, the anticommutation relations of the Θ^i fermions take the form $\{\Theta^{ia}_b, \Theta^{ia}_b\}_+ \sim g\mathcal{H}_4^{-1/4}\lambda^{-3/2}$. However, the rescaled fields are $\tilde{\Theta}^i = \lambda^{3/4}\mathcal{H}_4^{1/8}(gN)^{-1/2}\Theta^i$, so we have $\{\tilde{\Theta}^{ia}_b, \tilde{\Theta}^{ia}_b\} \sim \frac{1}{N}$. Normalization of $|0\rangle$ and the property that $\langle vac|\Theta^{i_1a_1}_{b_1} \cdots \Theta^{i_ka_k}_{b_k}|vac\rangle$ vanishes unless the fermions $\Theta^{i_1a_1}_{b_1} . \cdots \Theta^{i_ka_k}_{b_k}$ occur in pairs then tells us that the tensors $A^\beta_{i_1,\cdots,i_\beta}$ are independent of N to leading order in N.

Let us now briefly review the results of the purely bosonic argument above. At order k in perturbation theory, we find $2k$ factors of NL from the explicit factors of N in δH and from the energy denominators in P_{free}. In addition, each boson contributes at most one factor of L and a factor of N from summing over an SU(N) index.

Adding fermions introduces factors of $\tilde{\Theta}^i$ into the matrix elements. Each $\tilde{\Theta}^i$ introduces an additional sum over N, just as occurred for each boson. However, factors of $\tilde{\Theta}^i$ are not associated with factors of L since the infrared regularization affects only the bosonic zero modes. Consider a given term that is a product of matrix elements of the form (A.11), perhaps with additional fermion factors. Of course, two of the mode states appearing in each factor must in fact be the vacuum state $|0\rangle$.

We may write the $|0\rangle$ states in terms of $|vac\rangle$ by using (A.15). The fact that the free system is a direct product of boson and fermion degrees of freedom may be used to write each product of matrix elements in terms of two factors. One of these factors involves only the bosonic degrees of freedom and is much like (A.11) above. The other is just the expectation value of a product of fermion operators $\tilde{\Theta}^{ia_1}_{a_2}$ in the state $|vac\rangle$. The relevant point is that such expectation values vanish unless each $\tilde{\Theta}^{ia_1}_{a_2}$ that occurs in the product matches with some other fermion operator $\tilde{\Theta}^{jb_1}_{b_2}$; that is, the fermion operators must combine in pairs and create a factor setting $(a_1, a_2) = (b_1, b_2)$.

This introduces two constraints which may remove up to two sums over SU(N) indices. The first time a fermion is paired within a single trace, two new constraints are in fact imposed. Typically, however, one of the SU(N) indices on the $\tilde{\Theta}^{ia_1}_{a_2}$ factor will already have been fixed by a previous constraint. Therefore, in general we should only count this constraint as removing a single factor of N from our counting. Thus, these constraints remove one factor of N for each pair of fermions.[7]

If in addition we recall that with the current normalizations we have $\{\tilde{\Theta}^i, \tilde{\Theta}^j\}_+ \sim 1/N$, then we see that the creation of each pair removes at least one sum over SU(N) indices and provides an additional factor of N^{-1}. Recall that there was originally one sum over SU(N) indices (and thus one potential factor of N) for each fermion and thus two factors for each pair. We therefore see that to leading order the fermions contribute $N^0 \sim 1$ and $L^0 \sim 1$. As a result, the leading factors of N continue to appear precisely paired with the factors of L and we may again argue that the result is independent of N in the limit of large L.

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[7] If a trace involves only fermions (and no bosons; i.e., the sort of term that appears in (A.15)), then when the last fermion from a given trace is paired it may be that all of the constraints are already in place. In this case, no new constraints are introduced. Since, however, this can occur no more than once in any trace, and since our scheme undercounted the constraints imposed when the first fermion from the trace was paired, it is safe to state that the number of constraints imposed is at least the number of pairs created.

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